RICCI'S PRINCIPAL DIRECTIONS FOR A RIEMANN SPACE AND THE EINSTEIN THEORY

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In 1904 Ricci (*Atti R. Ist. Veneto*, **62,** 1230) developed the idea of principal directions in a Riemann space of n-dimensions, and in doing so introduced the contracted curvature tensor, which is fundamental in the Einstein theory, and gave a geometrical interpretation to it. A space in which these principal directions are completely indeterminate may be thought of as possessing a homogeneous character. We derive Ricci's results by a slightly different method, and then show that the three types of space, chosen by Einstein in 1914, 1917 and 1919, as spaces free from matter are of this homogeneous character, and include all types of such spaces.

Consider a Riemann space V_n of n-dimensions with the linear element

$$ds^2 = g_{ij}\,dx^i\,dx^j \; (g_{ij} = g_{ji}). \tag{1}$$

The right-hand member represents the sum of terms as i and j take on the values $1, \ldots\ldots n$, in accordance with the usual convention in such formulas that any term represents a summation with respect to each letter which appears in it both as a subscript and a superscript.

Suppose that we have in V_n n orthogonal unit vectors and let λ^i_h ($i = 1, \ldots\ldots n$) denote the contravariant components of the vector (h). Then we have

$$g_{ij}\lambda^i_h\lambda^j_k = \delta_{hk}, \tag{2}$$

where

$$\delta_{hk} = \begin{matrix} 1 \text{ for } h = k \\ 0 \text{ for } h \neq k. \end{matrix} \tag{3}$$

If we take the surface consisting of the geodesics tangent at a point P to the pencil of directions determined by the lines of two congruences (h) and (k) through P, the gaussian curvature of this surface at P is given by

$$r_{hk} = R_{pq,rs}\,\lambda^p_h\,\lambda^q_k\,\lambda^r_h\,\lambda^s_k, \tag{4}$$

where $R_{pq,rs}$ is the Riemann tensor of the first kind. By definition r_{hk} is the Riemann curvature of V_n at P for the directions (h) and (k) (cf. Bianchi, **1,** 342).

Since the n vectors are mutually orthogonal, we have

$$\sum_k \lambda^q_k\lambda^s_k = g^{qs}, \tag{5}$$

where g^{qs} is the cofactor of g_{qs} in the determinant of the g's divided by the determinant. Hence from (4) we have

$$\sum_k r_{hk} = -R_{pr}\lambda_h^p\lambda_h^r, \tag{6}$$

where

$$R_{pr} = -g^{qs}R_{pq,rs}, \tag{7}$$

and consequently is the contracted curvature tensor. From (4) it follows that $r_{hh}=o$, and therefore

$$\rho_h = \sum_k r_{hk} = -R_{pr}\lambda_h^p\lambda_h^r \tag{8}$$

is the sum of the Riemann curvatures determined by (h) and each of $n-1$ directions orthogonal to (h). Ricci calls ρ_h the *mean curvature* of V_n for the direction (h) at the point. Thus Ricci not only obtained the fundamental contracted tensor in 1904, but gave a geometrical interpretation of it.

In general as the vector (h) is changed the value of ρ_h varies. Since the components λ_h^p are bound by the equations $g_{pr}\lambda_h^p\lambda_h^r = 1$ in order to find the directions giving the maximum and minimum values of ρ_h, we equate to zero the derivatives with respect to λ_h^r $(r=1,\ldots\ldots n)$ of

$$\rho_h = -\frac{R_{pr}\lambda_h^p\lambda_h^r}{g_{pr}\lambda_h^p\lambda_h^r}. \tag{9}$$

This gives

$$(R_{pr} + \rho_h g_{pr})\lambda_h^p = 0. \tag{10}$$

Hence the maximum and minimum values of ρ_h are the roots of the equation

$$|R_{pr} + \rho g_{pr}| = 0, \tag{11}$$

and the direction for each ρ is given by the corresponding n equations (10) for $r=1,\ldots\ldots n$. Following Ricci we call these n directions the *principal directions* of the V_n. It is readily shown that if the roots of (11) are distinct, the n corresponding directions are mutually orthogonal.

If $\lambda_{h|p}$ denote the covariant components of (h), and we multiply (10) by $\lambda_{h|p}$ and sum for h, we have

$$R_{pr} = -\sum_h \rho_h\lambda_{h|p}\lambda_{h|r},$$

which is the form given by Ricci as characteristic of principal directions.

In order that the principal directions be completely indeterminate at every point, it is necessary that the coefficients of the λ's in (10) be zero; that is

$$R_{pr} = \varphi g_{pr}, \tag{12}$$

where φ is a scalar. When this is satisfied, we have from (9) that ρ_h is the same for all directions. Consequently the space may be thought of as homogeneous, and (12) is the necessary and sufficient condition for such homogeneity.

From (12) we have

$$R = g^{pr}R_{pr} = n\varphi,$$

and (12) becomes

$$R_{pr} = \tfrac{1}{n}\, g_{pr}R. \tag{13}$$

The original Einstein equations (1914) for space free from matter are those for which $\varphi = 0$ in (12); in 1917 (*Sitz. Pr. Ak. Wiss.*, Feb. 8), those for which $\varphi =$ const.; and in 1919 (*Ibid.*, Apr. 10), the general case (13) of a homogeneous space from the above point of view.

From (8), (9) and (12) it follows that for any three mutually orthogonal directions in any 3-space, which is homogeneous,

$$r_{12} + r_{13} = r_{21} + r_{23} = r_{31} + r_{32} = -\varphi,$$

whence

$$r_{12} = r_{13} = r_{23} = -\tfrac{\varphi}{2}.$$

Thus the Riemann curvature at each point is the same for all directions, and by the theorem of Schur (*Math. Ann.*, **27,** 563) is constant. Consequently the first type of Einstein space is a generalization of euclidean 3-space, and his other two spaces of 3-space of constant curvature.

NOTE ON THE DEFINITION OF A LINEAL FUNCTIONAL

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A linear functional is usually defined as one that is distributive and continuous, but the term continuous functional has been used in at least two ways which are not equivalent. F. Riesz, Fréchet, G. C. Evans and others have defined a continuous functional as one which satisfies the equation

$$\lim_{n \to \infty} L(u_n(x)) = L(u(x)), \tag{1}$$

when the sequence $u_1(x)$, $u_2(x)$, ... approaches $u(x)$ uniformly, while Levy[1] and W. L. Hart[2] have simply assumed that the sequence of u_n's converge in the mean. In Levy's paper the functional also depends on a parameter which has the same range as x, and $L(u_n(x),y)$ is only required to converge to $L(u(x),y)$ in the mean, but such a parameter will not be introduced here.

In what follows a distributive functional will be called linear when equation (1) is satisfied for uniformly convergent sequences of u's, and linear-*m* when they are only required to converge in the mean. If then L is to be linear it is necessary and sufficient that the equation

$$L(au_1 + bu_2) = aL(u_1) + bL(u_2) \tag{2}$$